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A PLANE PROBLEM OF MAGNETO-ELASTICITY FOR A FERROMAGNETIC MEDIUM WEAKENED BY CUTS†

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A plane problem of magneto-elasticity is considered for a soft ferromagnetic medium weakened by several cracks, which are treated as mathematical cuts in the initial (non-deformed) state. Since the approach used to analyse the effect of magnetization of the medium on the stress state in the neighbourhood of a rectilinear crack [1–3] is inapplicable in the general case, a new procedure is developed. It is based on representing the mechanical and magnetic quantities in terms of arbitrary analytic functions. The boundary-value problem is reduced to a system of two singular real integral equations of the first kind. Formulae for the stress intensity factors at the crack tip are obtained. An example is presented.

1. FORMULATION OF THE PROBLEM

CONSIDER an unbounded soft ferromagnetic medium in a rectangular Cartesian system of coordinates $Ox_1x_2x_3$, the medium being weakened by tunnel-shaped cuts L_j $(j=1, 2, \ldots, k)$ along the x_3 axis. We shall assume that the original magnetic field (in which the ferromagnetic body has been placed) is uniform over the whole space and directed parallel to the x_2 axis (Fig. 1). Because the material becomes magnetized, the body acquires a magnetic moment and subject to mechanical impact by the external field. We assume that mechanical loads (X_{1n}, X_{2n}, O), where $X_{in}=X_{in}(x_1, x_2)$, can act on the surface of the cracks. Under the influence of all these forces, the medium undergoes deformations, giving rise to an additional (induced) magnetic field. The problem consists of determining the mechanical and magnetic fields interacting with each other in the ferromagnetic medium with cracks.

In studies of this kind one usually uses a version of the theory based on the linearization of the equations of magneto-elasticity for soft ferromagnetic media (in the unsaturated state), neglecting magnetostriction and the effect of induction currents [1, 4, 5].

Suppose that the deformation of the body caused by the field of an infinitesimally small displacement vector (u_1, u_2, u_3) gives rise to small variations

$$B=B_0+b, \quad M=M_0+m, \quad H=H_0+h \tag{1.1}$$

of the original magnetic field. Here $B_0=(B_{0i})$, $H_0=(H_{0i})$, and $M_0=(M_{0i})$ $(i=1, 2, 3)$ are the magnetic induction, the magnetic field strength, and the magnetization corresponding to the non-deformed state of the body b, h, and m being the fluctuations of the above-mentioned quantities, which, by assumption, have the same order of magnitude as the elastic displacement vector.

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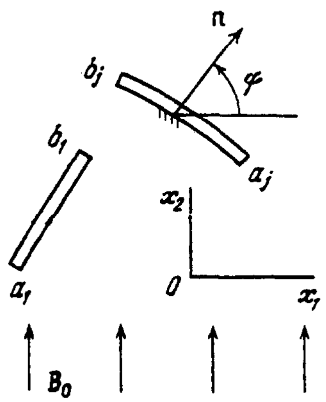

FIG. 1.

Then, provided that $|M_{0j}\partial_j u_i| \ll |m_i|$, we have the following system of linear relationships: the field equations (summation over repeated indices)

$$\partial_j t_{ji} + \mu_0(M_{0j}\partial_j H_{0i} + M_{0j}\partial_j h_i + m_j\partial_j H_{0i}) = 0 \tag{1.2}$$

$$\operatorname{rot} h = 0, \quad \operatorname{div} b = 0, \quad \partial_i = \partial / \partial x_i \quad (i, j = 1,2,3)$$

the material equations

$$m = \chi h, \quad b = \mu_0(h + m) = \mu_0\mu_r h, \quad \mu_r = 1 + \chi \tag{1.3}$$

$$t_{ij} = \sigma_{ij} + \chi\mu_0(H_{0i}H_{0j} + H_{0i}h_j + H_{0j}h_i)$$

$$T_{ij} = H_{0j}B_{0i} + H_{0j}b_i + B_{0i}h_j - \frac{1}{2}\mu_0(H_{0k}H_{0k})\delta_{ij} - \mu_0(H_{0k}h_k)\delta_{ij}$$

$$\sigma_{ij} = \lambda\delta_{ij}\partial_k u_k + \mu(\partial_j u_i + \partial_i u_j)$$

and the boundary conditions at the interfaces between the media

$$(S_{ij} - S_{ij}^e)n_j = 0, \quad S_{ij} = t_{ij} + T_{ij} \quad (i, j = 1,2,3) \tag{1.4}$$

$$\varepsilon_{ijk}\{n_j(h_k - h_k^e) - (H_{0k} - H_{0k}^e)n_m\partial_j u_m\} = 0$$

$$n_i(b_i - b_i^e) - (B_{0i} - B_{0i}^e)n_m\partial_i u_m = 0$$

Here t_{ij} and T_{ij} are the magneto-elastic and Maxwell stress tensors, ε_{ijk} is the permutation tensor, superscript e indicates that the corresponding physical quantity characterizes the external medium (in the case under consideration, the crack cavity), n_i is the projection of the unit vector normal to the interface onto the x_i axis, $\mu_0 = 4\pi \times 10^{-7}$ H/m is the absolute magnetic permeability, μ_r is the relative magnetic permeability, χ is the magnetic susceptibility of the medium, and λ and μ are the Lamé constants.

Relations (1.2)–(1.4) involve the components of the unperturbed magnetic field, which can be determined from the solution of the problem of magnetostatics for adjoining media (one of which is usually associated with vacuum)

$$\operatorname{rot} H_0 = 0, \quad \operatorname{div} B_0 = 0, \quad B_0 = \mu_0(H_0 + M_0) = \mu_0\mu_r H_0 \tag{1.5}$$

$$\operatorname{rot} H_0^e = 0, \quad \operatorname{div} B_0^e = 0, \quad B_0^e = \mu_0 H_0^e$$

$$\varepsilon_{ijk} \; n_j(H_{0k} - H_{0k}^e) = 0, \quad n_i(B_{0i} - B_{0i}^e) = 0$$

The system of equations (1.1)–(1.5) is complete in the sense that it enables one to determine all mechanical and magnetic quantities within the body, as well as the magnetic field in the surrounding medium.

Proceeding to the solution of the two-dimensional limiting problem for a ferromagnetic medium, we assume that the magnetic field B_0 is not affected by the presence of cracks, the latter being treated as mathematical cuts in the non-deformed state. Hence, we set $B_0 = (0, B_{02}, 0)$, where $B_{02} = \text{const}$. Equations (1.5) will obviously be satisfied for the following values of the magnetic field on the axis of a cut

$$H_{01}^e = \chi H_{02} n_1 n_2, \quad H_{02}^e (1+\chi n_2^2), \quad n_1 = \cos\psi, \quad n_2 = \sin\psi \tag{1.6}$$

where ψ is the angle between the normal to the left edge L_j (when going from the starting point a_j to the end point b_j) and the x_1 axis (Fig. 1).

In view of the fact that t_{ij}, T_{ij}, h_i are independent of x_3, relations (1.2), (1.3) yield [1]

$$\begin{aligned} &\nabla^2 u_i + \sigma\partial_i\vartheta + 2\chi\mu^{-1}\mu_0 H_{02}\partial_2 h_i = 0 \quad (i = 1,2) \\ &\nabla^2\Psi = 0, \quad h = \operatorname{grad}\Psi, \quad \vartheta = \partial_m u_m \\ &\sigma = (1-2\nu)^{-1}, \quad \nabla^2 = \partial_1^2 + \partial_2^2 \end{aligned} \tag{1.7}$$

The boundary conditions follow from (1.4). Taking (1.6) into account, we have

$$\begin{aligned} &n_1(b_1^\pm - b_1^e) + n_2(b_2^\pm - b_2^e) = \mu_0 M_{02} n_1 (U_1^*)^\pm \\ &n_1(h_2^\pm - h_2^e) - n_2(h_1^\pm - h_1^e) = M_{02} n_2 (U_1^*)^\pm \\ &U_1^* = n_m \partial u_m / \partial s, \quad \partial/\partial s = n_1\partial_2 - n_2\partial_1 \end{aligned} \tag{1.8}$$

where $h_i^\pm$, $b_i^\pm$, $(U_1^*)^\pm$ are the values of the corresponding quantities on L_j, and h_i^e and b_i^e are their values inside the "crack cavity" (on its axis).

From the four equations (1.8) we find the components of magnetic field fluctuations within the crack cavity

$$\begin{aligned} &h_1^e = (\chi n_1^2 + 1)h_1^\pm + n_1 n_2 m_2^\pm - M_{02}(U_1^*)^\pm \cos 2\psi \\ &h_2^e = n_1 n_2 m_1^\pm + (\chi n_2^2 + 1)h_2^\pm - M_{02}(U_1^*)^\pm \sin 2\psi \end{aligned} \tag{1.9}$$

Obviously, formulae (1.9) are meaningful under the following compatibility conditions for system (1.8)

$$\begin{aligned} &(\chi n_1^2 + 1)[h_1] + n_1 n_2 \chi [h_2] = M_{02} U_1 \cos 2\psi \\ &n_1 n_2 \chi [h_1] + (\chi n_2^2 + 1)[h_2] = M_{02} U_1 \sin 2\psi, \quad U_1 = [U_1^*] \end{aligned} \tag{1.10}$$

where the square brackets denote the jump of the corresponding quantity as the cut is crossed.

We obtain the mechanical boundary conditions on the edges L_j using (1.3), (1.6) and (1.7). We express them as complex equalities

$$\begin{aligned} &(t_{11} + t_{22})^\pm - e^{2i\psi}(t_{22} - t_{11} + 2it_{12})^\pm + 2\mu_0 M_{02}(H_{02}\alpha(\psi)U_1^* - n_2 n_m \chi h_m)^\pm = \\ &= 2(N - iT)^\pm + \mu_0 M_{02}^2 n_2^2, \quad \alpha(\psi) = \chi n_2 e^{-i\psi} - i \end{aligned} \tag{1.11}$$

Here N and T are the normal and shear forces acting at the edges L_j, the upper sign referring to the left edge (Fig. 1).

Below we shall assume that $[N-iT]=0$. It suffices to ensure that boundary equality (1.11) is satisfied at one of the edges of each crack only, provided we take into account the condition of continuous extendibility of its left-hand side as L_j is crossed

$$[t_{11}+t_{22}]-e^{2i\psi}[t_{22}-t_{11}+2it_{12}]+2\mu_0\chi H_{02}^2\beta(\psi)U_1=0 \tag{1.12}$$

$$\beta(\psi)=\chi n_1 n_2 \mu_r^{-1}-i(1+\chi n_2^2)$$

It follows that the boundary-value problem consists of finding a solution of the differential equations (1.7) that satisfies conditions (1.10) and (1.12) for the jumps and one of the boundary conditions (1.11) on L_j.

2. THE GENERAL SOLUTION OF SYSTEM (1.7)

One can see immediately that the volume expansion ϑ is a harmonic function. In view of this, we set

$$\mu\vartheta=(\kappa-1)\operatorname{Re}\Phi(z),\quad \Psi=\operatorname{Re}\{if(z)\},\quad \kappa=3-4\nu \tag{2.1}$$

where $\Phi(z)$ and $f(z)$ are arbitrary functions of the complex variable $z=x_1+ix_2$ analytic in the domain occupied by the body, and ν is Poisson's ratio.

Writing the equilibrium equations (1.7) as an equivalent complex equation with respect to u_1+iu_2 and integrating it in accordance with the definition (2.1) of volume expansion, we find that

$$2\mu(u_1+iu_2)=\kappa\varphi(z)-z\overline{\Phi(z)}-\overline{\varphi_1(z)}+\mu_0 M_{02}(z\overline{F(z)}-f(z)) \tag{2.2}$$

$$\Phi(z)=\frac{d\varphi}{dz}=\varphi'(z),\quad F(z)=f'(z)$$

where $\varphi_1(z)$ is an arbitrary analytic function and $\bar{f}$ is the complex conjugate function to f. From (2.1) and (2.2), taking (1.3) and (1.4) into account, we deduce that

$$\begin{aligned}
&t_{11}+t_{22}=4\operatorname{Re}\Phi(z)-2\mu_0 M_{02}\operatorname{Re}F(z)+\chi\mu_0 H_{02}^2\\
&t_{22}-t_{11}+2it_{12}=2\{\bar{z}W'(z)+W_1(z)\}+\chi\mu_0 H_{02}^2\\
&S_{11}+S_{22}=4\operatorname{Re}W(z)+2\chi\mu_0 H_{02}^2,\quad h_1-ih_2=iF(z)\\
&\Phi_1(z)=\varphi_1'(z),\quad W(z)=\Phi(z)-\mu_0 M_{02}F(z),\quad W_1(z)=\Phi_1(z)-\mu_0 M_{02}F(z)\\
&S_{22}-S_{11}+2iS_{12}=2\{\bar{z}W'(z)+W_1(z)\}-2B_{02}F(z)+(1+2\chi)\mu_0 H_{02}^2
\end{aligned} \tag{2.3}$$

It follows that the components of the magneto-elastic and Maxwell stress tensors as well as the mechanical displacement and magnetic field vectors in the ferromagnetic body can be expressed in terms of three arbitrary analytic functions. For $B_{02}=0$, formulae (2.2) and (2.3) are the same as the classical representations of the plane problem of the theory of elasticity. Some forms of representing the solutions of magneto-elasticity problems in terms of functions of a complex variable can be found in [6, 7].

We complete the formulation of the boundary-value problem by recasting conditions (1.9), (1.11), and (1.12) in terms of analytic functions. We have

$$[F(\zeta)]=M_{02}\delta(\psi)U_1(\zeta),\quad \zeta\in L=\bigcup_{j=1}^{k}L_j$$

$$\mathrm{Re}[\phi(\zeta)] - e^{2i\psi}[\zeta W'(\zeta) + W_1(\zeta)] + \chi\mu_0 H_{02}^2 \beta(\psi) U_1(\zeta) + \mathrm{Re}[W(\zeta)] = 0 \tag{2.4}$$

$$2\,\mathrm{Re}\,W(\zeta) - e^{2i\psi}\{\bar{\zeta} W'(\zeta) + W_1(\zeta)\} + \mu_0 M_{02}\Lambda = R(\zeta)$$

$$\Lambda = \mathrm{Im}\{F(\zeta)\overline{\alpha(\psi)}\} + H_{02}\alpha(\psi)U_1^*,$$

$$R(\zeta) = N - iT + \frac{1}{2}\chi\mu_0 H_{02}^2(\chi n_2^2 + e^{2i\psi} - 1)$$

$$2\mu U_1^*(\zeta) = \mathrm{Im}\{\mu_0 M_{02} F(\zeta) - W(\zeta) - \kappa\phi(\zeta) + e^{2i\psi}(\bar{\zeta} W'(\zeta) + \phi_1(\zeta))\}$$

3. INTEGRAL REPRESENTATIONS OF THE SOLUTIONS

Following [8], the representations of the desired analytic functions, which are correct in the sense that the conditions for the jumps in (2.4) are satisfied independently of the densities occurring in them, will be written in the form

$$\begin{gathered}
F(z) = \frac{\mu_0 M_{02}}{2\pi i}\int_L \frac{\omega(\zeta)}{\zeta - z}d\zeta, \quad W(z) = -\frac{1}{2\pi i}\int_L \frac{p_1(\zeta)}{\zeta - z}d\zeta \\
W_1(z) = \frac{1}{2\pi i}\int_L \frac{\bar{\zeta} p_1(\zeta)}{(\zeta - z)^2}d\zeta + \frac{1}{2\pi i}\int_L \frac{\overline{p_2(\zeta)}}{\zeta - z}d\bar{\zeta} \\
\Phi(z) = W(z) + \mu_0 M_{02} F(z), \quad \Phi_1(z) = W_1(z) + \mu_0 M_{02} F(z) \\
p_j(\zeta) = \frac{2\mu}{\kappa + 1}\{iU_1(\zeta) - U_2(\zeta) + e_0^2 \Lambda_j(\psi) U_1(\zeta)\} \quad (j = 1, 2) \\
\Lambda_1(\psi) = \frac{\chi}{2}\left\{i + \chi e^{-i\psi}\left(\frac{in_1}{\mu_r}(2 - \kappa) - (\kappa + 1)n_2\right)\right\} \\
\Lambda_2(\psi) = \frac{i\chi}{2}\left\{-\kappa + \chi e^{-i\psi}(2 - \kappa)\frac{n_1}{\mu_r}\right\}, \quad e_0^2 = \frac{\mu_0 H_{02}^2}{\mu}
\end{gathered} \tag{3.1}$$

The densities U_j can be expressed in terms of the displacements by the formula (ds being the element of arc of L)

$$(U_1 + iU_2)e^{i\psi} = d/ds[u_1 + iu_2] \tag{3.2}$$

which is consistent with the definition of U_1 in (1.8) and (1.10).

The functions (3.1) do not yet fully correspond to the physical content of the boundary-value problem in question. One has to ensure that the single-valuedness conditions for the displacements are satisfied in the domain occupied by the medium, and also that the equalities

$$\int_{C_j} (n_1 b_1 + n_2 b_2)ds = 0, \quad \int_{C_j} (n_1 h_2 - n_2 h_1)ds = 0 \quad (j = 1, 2, \ldots, k) \tag{3.3}$$

are satisfied, C_j being an arbitrary closed contour encompassing L_j. By (3.1), the single-valuedness conditions for the displacements can be reduced to

$$\int_{L_j} (U_1 + iU_2)d\zeta = 0 \quad (j = 1, 2, \ldots, k) \tag{3.4}$$

Taking (2.3) into account, we can represent (3.3) in the equivalent form

$$\int_{L_j} \omega(\zeta)d\zeta = 0 \quad (j = 1, 2, \ldots, k) \tag{3.5}$$

Hence the integral representations (3.1) make sense only if conditions (3.4) and (3.5) are satisfied. In fact, equalities (3.5) follow from (3.4). This can easily be seen by invoking the first formula in (2.4) and the definition (3.2).

4. INTEGRAL EQUATIONS OF THE BOUNDARY-VALUE PROBLEM (2.4)

Substitution of the boundary values of the functions (3.1) into boundary condition (2.4) at one of the edges of L leads to the following system of singular integral equations of the first kind

$$\sum_{n=1}^{2} \int_L U_n(\zeta)dH_{mn}(\zeta,\zeta_0) = N_m(\zeta_0), \quad \zeta_0 \in L \quad (m=1,2) \tag{4.1}$$

$$4dH_{11} = \operatorname{Im}\{[2H_2H_3^0 g - 4H_1(\kappa+1)H_4^0 - 2H_1H_3^0 h - i\chi^2 e_0^2(\kappa+1)\delta(\psi)H_s^0]d\tau\},$$

$$d\tau = (\zeta-\zeta_0)^{-1}d\zeta$$

$$4dH_{12} = \operatorname{Im}\{[4 + i\chi^2 e_0^2(\kappa+1)n_1^0 n_2^0 + 2H_3^0(h-g)]d\tau\}$$

$$4dH_{21} = \operatorname{Re}\{[2H_6^0(H_1 h - H_2 g) + \chi(1+\chi\sin^2\psi_0)e_0^2(\kappa+1)H]d\tau\}$$

$$4dH_{22} = \operatorname{Re}\{[2H_6^0(g-h) - \chi(1+\chi\sin^2\psi_0)e_0^2(\kappa+1)]d\tau\}$$

$$4N_1(\zeta) = \pi(\kappa+1)\left[\frac{2N}{\mu} + \chi e_0^2(\chi n_2^2 - 1 + \cos 2\psi)\right]$$

$$2N_2(\zeta) = \pi(\kappa+1)\left[\chi e_0^2 n_1 n_2 - \frac{T}{\mu}\right], \quad g = e^{2i(\psi_0-\psi)}$$

$$h = e^{2i\psi_0}\frac{\bar\zeta-\bar\zeta_0}{\zeta-\zeta_0}, \quad H_j = H_j(\psi), \quad H_j^0 = H_j(\psi_0)$$

$$H_1 = i + e_0^2\Lambda_1(\psi), \quad H_2 = e_0^2\overline{\Lambda_2(\psi)} - i, \quad H_3 = 2H_4 + \frac{\kappa-1}{\kappa+1}$$

$$H_4 = \frac{1}{\kappa+1} + i\chi^2 e_0^2\frac{n_1 n_2}{4}$$

$$H_6 = 1 + \chi(1+\chi n_2^2)\frac{e_0^2}{2}$$

$$H_s = \overline{\alpha(\psi)} + \chi^2 e_0^2(1-\kappa+e^{2i\psi})\frac{n_1 n_2}{2}$$

$$H(\psi,\psi_0) = H_1 + \frac{1}{2}\chi^2 e_0^2\delta(\psi)(1-\kappa+e^{2i\psi_0})$$

This system must be considered together with additional conditions (3.4). Moreover, provided that the contours L_j have no common points and their curvatures and the functions $N_i(\zeta)$ satisfy the Hölder condition, we obtain a unique solution in the class of functions unbounded at the end-points of L_j.

5. THE STRESS INTENSITY FACTORS

We will introduce the following parametric representation of L_j (subscript j will be omitted below): $\zeta = \zeta(\beta)$, where $-1 \leqslant \beta \leqslant 1$. Correspondingly, we set

$$U_m(\zeta)=\Omega_m(\beta)/\sqrt{1-\beta^2} \tag{5.1}$$

where $\Omega_m(\beta)$ satisfies the Hölder condition in $[-1, 1]$.

On introducing the function (5.1) into the integral representations (3.1), as a result of the standard procedure for separating the singularities of the solution at the end-points of L_j, we find that

$$K_{\rm I}^s-iK_{\rm II}^s=\pm\frac{2\mu\sqrt{s'(\pm1)}}{\kappa+1}\left\{i\Omega_2(\pm1)-\Omega_1(\pm1)+\frac{\chi}{4}e_0^2\Omega_1(\pm1)\left(-2+\sum_{\alpha,\delta=1}^{2}a_{\alpha\delta}n_\alpha^\pm n_\delta^\pm\right)\right\},$$

$$n_1^\pm=\cos\psi(\pm1),\quad n_2^\pm=\sin\psi(\pm1)$$

$$K_{\rm I}^s-iK_{\rm II}^s=\sqrt{2r}(S_{nn}-iS_{ns}),\quad r=|z-c|\to0$$

$$a_{11}=\frac{2\chi(\kappa-2)}{\chi+1},\quad a_{22}=-2\chi(\kappa+1)\quad a_{12}=a_{21}=i\chi\left(\kappa+1+\frac{\kappa-2}{\chi+1}\right)$$

where the upper sign refers to the tip $c=b$ ($\beta=1$), while the lower sign refers to $c=a$ ($\beta=-1$), S_{nn} and S_{ns} being the normal and tangential total stresses over an area with normal vector $n=(\cos\psi_c,\ \sin\psi_c)$ in a region extending beyond the tip c.

6. EXAMPLE

Consider a medium with a rectilinear crack of length $2l$ oriented relative to the original magnetic field as in Fig. 2(a). We take the parametric equation to be $\zeta=il\beta e^{i\psi}$, $\zeta_0=il\beta_0e^{i\psi}$, where $-1\leqslant\beta$, $\beta_0\leqslant1$. This being the case, the integral equation (4.1) and the additional condition (3.4) take the form

$$\sum_{n=1}^{2}p_{mn}\int_{-1}^{1}\frac{\Omega_n(\beta)d\beta}{\sqrt{1-\beta^2}(\beta-\beta_0)}=N_m^*(\beta_0)\quad(m=1,2) \tag{6.1}$$

$$\int_{-1}^{1}\frac{\Omega_1(\beta)+i\Omega_2(\beta)}{\sqrt{1-\beta^2}}d\beta=0,\quad N_m^*(\beta)=N_m(\zeta)$$

$$p_{11}=\sum_{j=0}^{4}d_j\chi^j,\quad p_{21}=\sum_{j=2}^{3}d_j^*\chi^j,\quad p_{12}=\chi^2e_0^2(\kappa-1)\frac{n_1n_2}{4}$$

$$4p_{22}=4+\chi e_0^2(1-\kappa)(1+\chi n_2^2),\quad d_0=-1,\quad 4d_1=(\kappa-1)e_0^2$$

$$4d_2=-\frac{n_1^2}{\mu_r}(5-\kappa)e_0^2,\quad 4d_3=(\kappa+1)e_0^2n_2^2,\quad n_1n_2d_3^*=-(1+\chi n_2^2)d_4,$$

$$4d_4=e_0^4n_1^2n_2^2\left(\kappa+1+\frac{1-2\kappa}{\mu_r}\right)$$

$$4d_2^*=e_0^2n_1n_2\left(\kappa+1+\frac{2\kappa-4}{\mu_r}\right)$$

The solution of the characteristic system (6.1) is

$$\Omega_n(\beta)=A_n\beta\quad(n=1,2) \tag{6.2}$$

$$\pi\Delta A_1=p_{22}N_1^*-p_{12}N_2^*,\quad \pi\Delta A_2=p_{11}N_2^*-p_{21}N_1^*$$

$$\Delta=p_{11}p_{22}-p_{12}p_{21}$$

Correspondingly, one can determine the desired analytic functions and their combinations (Fig. 2b) to be

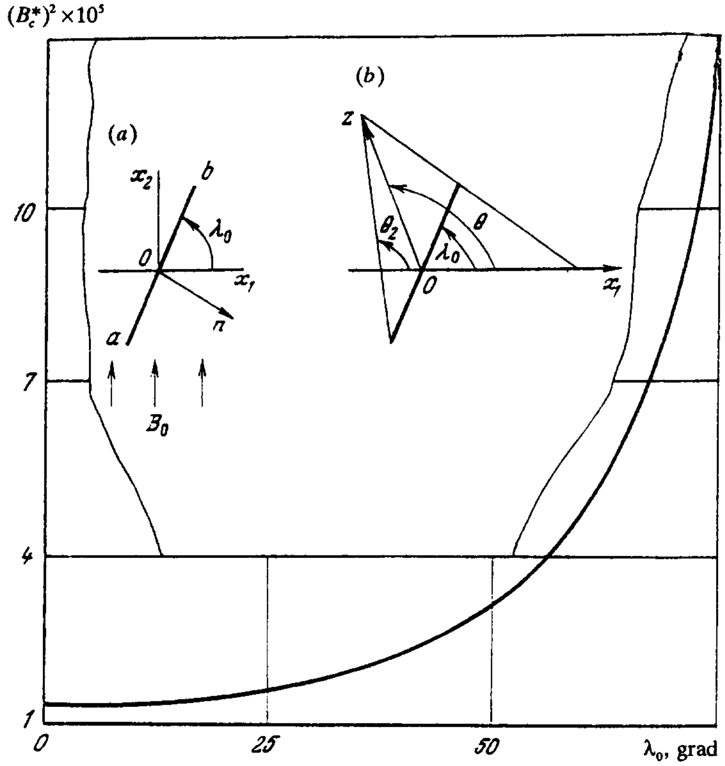

FIG. 2.

$$F(z)=\frac{1}{2i}M_{02}A_1\delta(\psi)\gamma_1(z),\quad W(z)=i\mu\bar{\varepsilon}_1\gamma_1(z) \tag{6.3}$$

$$W_1(z)=i\mu e^{-2i\psi}(\bar{\varepsilon}_1\gamma_2(z)+\bar{\varepsilon}_2\gamma_1(z)),\quad \lambda_0=\psi-\frac{3}{2}\pi$$

$$\gamma_1(z)=\frac{z}{\sqrt{z^2-c^2}}-1,\quad \gamma_2(z)=\frac{z^3-2zc^2}{\sqrt{(z^2-c^2)^3}}-1,\quad c=le^{i\lambda_0}$$

$$\bar{z}W'(z)+W_1(z)=\mu\left\{\frac{i\bar{\varepsilon}_1}{\sqrt{\rho_1\rho_2}}\sin(\theta_2-\theta_1)\exp\left[i\lambda_0-\frac{3i}{2}(\theta_1+\theta_2)\right]-\right.$$

$$\left.-i(\bar{\varepsilon}_1+\bar{\varepsilon}_2)\left[\frac{\rho}{\sqrt{\rho_1\rho_2}}\exp\left(i\theta-i\frac{\theta_1+\theta_2}{2}\right)-1\right]e^{-2i\lambda_0}\right\}$$

$$(\kappa+1)\varepsilon_1=A_1\overline{H}_1-A_2,\quad (\kappa+1)\varepsilon_2=A_1\overline{H}_2-A_2$$

$$z-c=\rho_1e^{i\theta_1},\quad z+c=\rho_2e^{i\theta_2},\quad z=\rho e^{i\theta}$$

Using (6.3), we can find the mechanical and magnetic quantities at any point of the body. In particular, on a straight line parallel to the crack we have

$$S_{nn}-iS_{ns}=2i\mu\varepsilon_1\left\{1-\frac{\rho}{\sqrt{\rho_1\rho_2}}\cos\left(\theta-\frac{\theta_1+\theta_2}{2}\right)\right\}+ \tag{6.4}$$

$$+\mu i\bar{\varepsilon}_1\frac{\sin(\theta_2-\theta_1)}{\sqrt{\rho_1\rho_2}}\exp\left\{3i\left(\lambda_0-\frac{\theta_1+\theta_2}{2}\right)\right\}+\mu e_0^2\left(\chi+\left(\chi+\frac{1}{2}\right)e^{2i\lambda_0}\right)$$

$$\varepsilon_1=\frac{i\chi e_0^2}{4(\kappa+1)}A_1\left(-2+\sum_{\alpha,\sigma=1}^{2}a_{\alpha\delta}n_\alpha n_\delta\right)-\frac{A_2+iA_1}{\kappa+1}$$

But if the straight line coincides with the crack axis, we find from (6.4) that

$$S_{nn}-iS_{ns}=\mu\left\{e_0^2\left(\chi+\left(\chi+\frac{1}{2}\right)e^{2i\lambda_0}\right)+2i\varepsilon_1\right\} \tag{6.5}$$

for $\rho\leqslant l$. If $\rho>l$, the additional term $-2i\varepsilon_1\mu\rho/\sqrt{(\rho^2-l^2)}$ will occur on the right-hand side of (6.5).

Using these relationships, one can observe the effect of the orientation of the crack with respect to the original magnetic field B_0 on the mechanical and magnetic quantities. The simplest results can be obtained for a horizontal or vertical crack. In this case, $n_1n_2=0$, $p_{12}=p_{21}=0$, and the system of equations (6.1) splits into two independent equations, the quantities A_n that occur in (6.2) being such that

$$\pi p_{11}A_1=N_1^*,\quad \pi p_{22}A_2=N_2^* \tag{6.6}$$

where

$$4p_{11}=\chi e_0^2\{(\kappa+1)\chi^2+(\kappa-1)\}-4,\quad N_2^*=-\frac{\pi T}{2\mu}(\kappa+1)$$

$$4p_{22}=4-\chi e_0^2\mu_r(\kappa-1),\quad N_1^*=\frac{\pi}{2}(\kappa+1)\left\{\frac{N}{\mu}+\frac{\chi}{2}e_0^2(\chi-2)\right\}$$

for a horizontal crack $(n_1=0,\ n_2=-1)$, and

$$4p_{11}=\chi e_0^2\left(\kappa-1-\chi\frac{5-\kappa}{\mu_r}\right)-4,\quad N_2^*=-\frac{\pi T}{2\mu}(\kappa+1)$$

$$4p_{22}=4-\chi e_0^2(\kappa-1),\quad N_1^*=\frac{\pi N}{2\mu}(\kappa+1)$$

for a vertical crack $(n_1=1,\ n_2=0)$

It follows immediately from (6.4)–(6.6) that there are values of $b_c^2=(\mu_r e_0)^2=B_{02}^2/(\mu\mu_0)$, which are called critical, for which the components of the magneto-elastic and Maxwell stress tensors as well as the magnetic field and induction vectors in a ferromagnetic medium with a crack become infinite. This effect was discovered in the case of a horizontal crack in [1–3]. At the same time, this is not so for a vertical crack (parallel to the original magnetic field). As follows from (6.6), there is no such critical value for $N\neq 0$ and $T=0$, while for $N=0$ and $T\neq 0$ it exists theoretically ($b_c^*=6.3\times 10^2$ when $\mu_r=10^5$ and $\nu=0.25$), but it cannot, in fact, be realized. Taking into account that the total stresses in a uniform ferromagnetic medium in the presence of the original magnetic field $(0,\ B_{02},\ 0)$ are $2S_{11}=-\mu_0H_{02}^2$ and $S_{12}=0$, $2S_{22}=(4\chi+1)\mu_0H_{02}^2$, one can conclude that the plane equilibrium form is impossible in a ferromagnetic medium with a crack if $b_c=b_c^*$.

For a rectilinear crack oriented at an angle $\pi/2-\lambda_0$ to the initial magnetic field, the critical value b_c^* can be defined in accordance with (6.2) as the least positive root of the equation $\Delta=0$. This is an algebraic equation of the fourth degree in b_c. The least value b_c^* is attained for $\lambda_0=0$ ($b_c^*=3.6\times 10^{-3}$ when $\mu_r=10^5$ and $\nu=0.25$). Furthermore, as λ_0 increases from 0 to $\pi/2$, b_c increases monotonically up to the value for a vertical crack specified above (Fig. 2).

It follows from the above that in the general situation (in the setting of [4]) a spectrum of critical values of b_c exists depending on the configuration of the cracks, their position with respect to one another, their orientation relative to the initial magnetic field, and the type of loads for which the plane equilibrium form is impossible.

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